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# SIMPLE SHEARING OF AN INCOMPRESSIBLE, VISCOELASTIC QUADRATIC MATERIAL

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**Abstract**—Viscoelastic constitutive equations of the differential type and of the rate type are applied in the study of the mechanical response of rubberlike materials in a simple shearing deformation superimposed on a specified static longitudinal stretch. Creep, recovery and stress relaxation processes are formulated for a class of incompressible, isotropic viscoelastic quadratic materials. Equations characterizing the creep and recovery processes for materials of differential type are derived, and their exact solutions are obtained. For materials of the rate type, the stress relaxation process is independent of the elastic material response functions. A coupled system of equations for the creep and recovery processees in a material of rate type are presented; these must be solved numerically. The finite amplitude, damped, free vibration of a rigid body supported by incompressible, viscoelastic quadratic simple shear mountings of the differential type is investigated. The motion of the body is governed by a damped Duffing equation whose solution is discussed by the averaging method.

## 1. INTRODUCTION

Creep, recovery and stress relaxation phenomena are commonly encountered in engineering applications that use elastomers or soft polymers. Typical applications of these rubberlike materials in mechanical systems include machine mountings, foundation springs and packaging supports designed to absorb and to control vibration. In some instances, due to impact or to a strong earthquake, for example, the system may experience large amplitude oscillatory motions and internal material damping provides a natural vibration absorber effect. To characterize effects of creep, recovery, stress relaxation and viscous damping in the study of mechanical systems that employ elastomeric springs, general constitutive equations for viscoelastic materials subjected to finite deformations must be introduced. For mathematical studies, the constitutive equation should cover a broad class of real materials, it should characterize the major nonlinear physical phenomena of interest, and it should be easy to apply.

These attributes have been the focus of recent studies of physically sound and mathematically simple constitutive equations of the differential type [see e.g. Beatty and Zhou (1991); Zhou (1991a)] and of the rate type [see e.g. Zhou (1991b)]. Beatty and Zhou (1991) introduced a class of incompressible, viscohyperelastic materials of differential type, a class of rubberlike materials that generalizes the Kelvin–Voigt solid of classical linear viscoelasticity, to study the quasi-static response of a rubberlike material in a simple shearing deformation superimposed on a given static homogeneous strain. The differential model delivers simple analytical solutions for the creep and recovery processes for the special class of viscoelastic Mooney–Rivlin materials. It also leads to a classical closed form solution to the problem of the finite amplitude, damped, free vibrations of a simple shear spring-mass system. The constitutive equation used by Beatty and Zhou (1991) is essentially a combination of finite elasticity theory and linear viscous fluids theory. It describes the uncoupled linear viscous response and nonlinear elastic response of an isotropic, incompressible material. Zhou (1991a) extended this model to include a nonlinear power law viscous fluid effect and a nonlinear second order fluid contribution. For a simple shearing deformation, the nonlinear models of differential type deliver exact solutions for creep and recovery processes. The two models studied by Zhou (1991a) include, as a special case, the viscoelastic constitutive model introduced by Beatty and Zhou (1991), and they generalize the Kelvin–Voigt solid and the three-parameter model of classical linear viscoelasticity.

The differential type theory involves no stress rate, therefore, the stress relaxation phenomenon was not addressed in these primary studies. To investigate this problem, Zhou (1991b) introduced an incompressible, viscoelastic constitutive equation of the rate type. This model includes the equation of differential type studied by Beatty and Zhou (1991), it generalizes the standard linear solid of classical viscoelasticity theory, and it predicts creep, recovery and stress relaxation effects. The stress relaxation process described by the rate type model is characterized by a solution that is independent of the elastic material response functions; it depends on a material retardation time constant only. Solutions for the simple shearing deformation of a viscoelastic Mooney–Rivlin material and a simple extension of a viscoelastic neo-Hookean material, both of the rate type, are provided. The analysis of the creep and recovery phenomena in a simple shearing deformation, however, is incorrect. These results will be reviewed and corrected below.

This paper is a continuation of work by Beatty and Zhou (1991) and Zhou (1991b). The incompressible, viscoelastic Mooney–Rivlin model introduced in these works is characterized by a constant shear response function. In consequence, the model extends classical exact solutions to creep, recovery and stress relaxation processes to a special class of nonlinear, viscohyperelastic materials. In the present study, however, the creep, recovery and stress relaxation processes are examined for a class of incompressible, isotropic viscohyperelastic quadratic materials in simple shear. Here we examine the effects of nonlinearity of the shear response functions of an incompressible, viscohyperelastic material whose elastic strain energy is a general quadratic function of the principal invariants of the left Cauchy–Green deformation tensor. The Mooney–Rivlin model is included as a special limit case for which the nonlinearity vanishes.

Constitutive equations for incompressible, isotropic viscoelastic materials of the differential and rate types are reviewed in Section 2, and the response functions for the class of quadratic materials are introduced there. A simple shearing deformation superimposed on a finite, static uniaxial deformation is considered in Section 3. Exact solutions are presented for the creep shearing and recovery of a viscoelastic quadratic material of differential type. It is seen that the nonlinearity increases the creep rate so that the material more quickly approaches its ultimate equilibrium shear state under a constant applied shear load. In recovery, however, when the shearing load is removed, the nonlinearity has only a slight effect on the recovery rate for a quadratic material of differential type. The stress relaxation process is studied for a quadratic material of the rate type. It is found that the process is characterized by an essentially universal solution, as shown by Zhou (1991b); only the material retardation time constant appears in the solution. A coupled system of equations for the creep shearing and recovery of general incompressible, viscoelastic materials of the rate type are derived. These equations may be studied numerically, but we do not pursue this here. Finally the problem of the finite amplitude, damped, free vibrations of a load supported by simple shear mountings is studied in Section 4 for the class of viscoelastic quadratic materials of differential type. The motion of the load is governed by a damped Duffing equation whose approximate solution is obtained by the averaging method.

## 2. CONSTITUTIVE EQUATIONS OF THE DIFFERENTIAL AND RATE TYPES

A constitutive equation for an incompressible, isotropic, nonlinear viscoelastic solid of differential type was introduced by Beatty and Zhou (1991) in their study of the quasi-static mechanical response of rubberlike materials in simple shear. This equation is given by

$$\mathbf{T} = -p\mathbf{1}+\beta_1\mathbf{B}+\beta_{-1}\mathbf{B}^{-1}+2\eta\mathbf{D}, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $p$ is an undetermined pressure due to the incompressibility constraint, $\beta_1$ and $\beta_{-1}$, the elastic response functions of the material, are functions of the principal invariants $I_1$ and $I_2$ of the left Cauchy–Green deformation tensor $\mathbf{B}$, $\eta \geqslant 0$ denotes the constant material viscosity, and $\mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^T)$ is the stretching tensor. In the last expression $\mathbf{L}$ denotes the spatial velocity gradient tensor. The material is isotropic relative to its natural undeformed state $\chi$, and all deformations are taken relative to $\chi$. Equation (1) is a composition of the constitutive equation for an incompressible nonlinear elastic solid and the constitutive equation for an incompressible linear viscous (Newtonian) fluid, and hence describes the uncoupled linear viscous response and the nonlinear elastic response of an isotropic, incompressible material. It is shown by Beatty and Zhou (1991) that this equation generalizes the Kelvin–Voigt solid of classical linear viscoelasticity.

A constitutive equation for an incompressible, isotropic, nonlinear viscoelastic solid of the rate type was introduced by Zhou (1991b) in the study of creep, recovery and stress relaxation processes in both simple shear and simple extension. This equation is given by

$$\mathbf{T} = -p\mathbf{1}+\beta_1\mathbf{B}+\beta_{-1}\mathbf{B}^{-1}+2\eta\mathbf{D}-\xi(\dot{\mathbf{T}}+\mathbf{L}^T\mathbf{T}+\mathbf{T}\mathbf{L}), \tag{2}$$

where $\xi$ is a certain positive material time constant and a superimposed dot denotes the material time derivative. It is evident that the rate type material (2) reduces to the differential type material (1) when $\xi \to 0$. As before, the material is isotropic relative to its undeformed state $\chi$, and all deformations are considered relative to $\chi$. It is shown by Zhou (1991b) that eqn (2) generalizes the standard linear solid model of classical linear viscoelasticity theory.

The elastic response functions $\beta_1$ and $\beta_{-1}$ for an incompressible hyperelastic material [see e.g. Beatty (1987)] are defined by

$$\beta_1 = 2\frac{\partial\Sigma}{\partial I_1}, \quad \beta_{-1} = -2\frac{\partial\Sigma}{\partial I_2}, \tag{3}$$

in which the strain energy density $\Sigma$ is a function of the principal invariants $I_1$ and $I_2$ of $\mathbf{B}$. In this work, we consider a class of incompressible quadratic materials [see e.g. Beatty (1984)] for which the strain energy is given by

$$\Sigma = C_1(I_1-3)+C_2(I_2-3)+C_3(I_1-3)^2+C_4(I_2-3)^2+C_5(I_1-3)(I_2-3), \tag{4}$$

in which $C_k$, $k = 1\ldots 5$, are material constants. In particular, when $C_3 = C_4 = C_5 = 0$, the material is a Mooney–Rivlin material, and when $C_2 = 0$ as well, the material is neo-Hookean. Thus, the response functions (3) for an incompressible, quadratic, hyperelastic material (4) are provided by

$$\begin{aligned} \beta_1 &= 2C_1+4C_3(I_1-3)+2C_5(I_2-3) \\ \beta_{-1} &= -2C_2-4C_4(I_2-3)-2C_5(I_1-3). \end{aligned} \tag{5}$$

A material characterized by eqns (1) or (2), respectively, in which the response functions are defined by eqn (5), is called a viscoelastic quadratic material of differential type or rate type. It is shown by Zhou (1991b) that the stress relaxation process for a material of the rate type is described by a universal solution regardless of the response functions in eqn (2). Hence, this universal solution is also valid for a quadratic material having the specific response functions (5). The universal solution will be reviewed later. We shall begin our study with a brief description of a simple shear support system and the principal invariants characteristic of a simple shear deformation superimposed on a static uniaxial stretch. We then describe the creep and recovery processes for the two types of viscoelastic quadratic materials introduced above.

## 3. CREEP AND RECOVERY OF VISCOELASTIC QUADRATIC MATERIALS IN SIMPLE SHEAR

In view of its mathematical simplicity and wide use in engineering applications, the simple shear deformation has received broad attention in the study of both mechanical design and vibration problems [see e.g. Beatty (1984), (1988), (1989); Beatty and Bhattacharyya (1989); Bhattacharyya (1989); Beatty and Zhou (1991)]. It was also used recently in the analysis of creep and recovery processes for viscohyperelastic materials of the differential and rate types [see e.g. Beatty and Zhou (1991); Zhou (1991a, b)]. Beatty and Zhou (1991) deal with a simple shear superimposed on a triaxial stretch, while Zhou (1991a, b) examines a simple shear superimposed on a simple longitudinal stretch. Here we follow the latter model and consider a rigid load of mass $M$ on a smooth inclined surface making an angle $\theta$ with the horizonal plane and supported symmetrically between identical, prestretched rubber springs of undeformed length $L$ and uniform cross-sectional area $A$. The springs, prestretched an amount $\lambda_s$, are bonded to the load at one end and to rigid end supports at the other, as shown in Fig. 1. We suppose that each rubber spring executes an ideal, time-dependent simple shear deformation of amount $K(t) = \tan\gamma(t)$ superimposed on the static longitudinal stretch $\lambda_s$. Clearly, the simple shear shown in Fig. 1 is an ideal deformation in which $\gamma(t)$ denotes the angle of shear, and bending of the springs is ignored. The principal invariants of $\mathbf{B}$ for the simple shear deformation relative to the natural, undeformed state, as shown by Beatty and Zhou (1991), are given by

$$I_1(\mathbf{B}) = \lambda_s^2(1+K^2)+2\lambda_s^{-1}, \quad I_2(\mathbf{B}) = \lambda_s(2+K^2)+\lambda_s^{-2}, \quad I_3(\mathbf{B}) = 1. \tag{6}$$

The two incompressible viscoelastic models characterized by eqns (1) and (2) will be used in the analysis of the simple shear deformation. The creep and recovery processes are considered next for each material model in turn.

### 3.1. *Creep and recovery of a material of differential type in simple shear*

For the viscoelastic material of differential type (1), it was shown by Beatty and Zhou (1991) that the Cauchy shear stress $T_{12}$ exerted on the mountings at the load interface, as illustrated in Fig. 1, is determined by

$$T_{12} = \lambda_s K\hat{\mu}(K^2;\lambda_s)+\eta\dot{K}, \tag{7}$$

in which

$$\hat{\mu}(K^2;\lambda_s) \equiv \lambda_s\beta_1-\beta_{-1} \tag{8}$$

defines the elastic shear response function for the material. Equation (7) shows that the

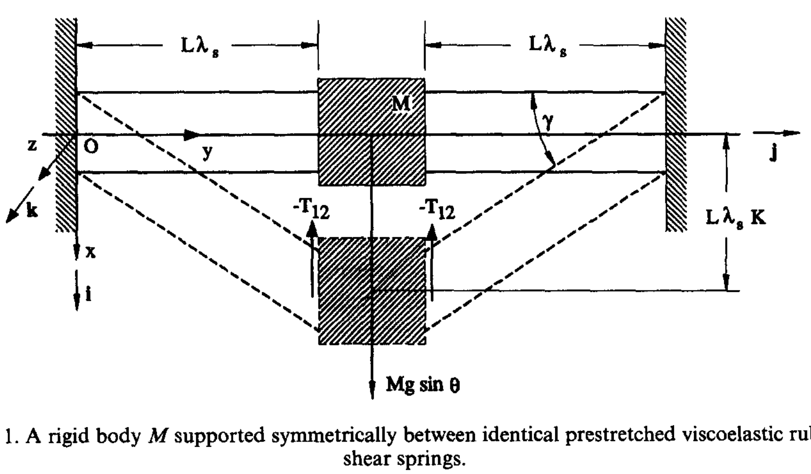



Fig. 1. A rigid body $M$ supported symmetrically between identical prestretched viscoelastic rubber shear springs.

shear stress is a function of both the amount of shear $K$ and its time rate of change $\dot{K}$, a relation typical of a classical Kelvin–Voigt material.

It is useful to recall the empirical inequalities for the elastic response functions [see e.g. Beatty (1987)], namely,

$$\beta_1 > 0, \quad \beta_{-1} \leqslant 0. \tag{9}$$

We require that these hold for all deformations of our materials. In consequence, for all values of $K \in (-\infty, \infty)$ and for each fixed stretch $\lambda_s \in (0, \infty)$, we see from eqn (8) that

$$\hat{\mu}(K^2; \lambda_s) > 0. \tag{10}$$

When $\lambda_s = 1$, we write

$$\hat{\mu}(K^2; 1) \equiv \mu(K^2) \quad \text{with} \quad \mu(0) \equiv \mu_0, \tag{11}$$

where $\mu_0$ is the usual elastic shear modulus of the natural, undeformed state.

The creep process in a simple shear deformation is characterized by the growth of $K(t)$ under a constant applied engineering stress, for example, $\hat{S}_{12}$. In this case, the corresponding Cauchy stress component $\hat{T}_{12} = \lambda_s \hat{S}_{12}$ is also constant, and in the ultimate static equilibrium configuration of the system, shown in Fig. 1, we have $2A\hat{S}_{12} = Mg_0$, where $g_0 \equiv g \sin \theta$. It is natural to expect that if the load is released from rest when $K = 0$, the shear will increase asymptotically to the ultimate equilibrium state defined by $\dot{K}(t) \to 0$ and $K(t) \to K_s$ as $t \to \infty$. Hence, the ultimate static equilibrium shear deflection $K_s$ is related to $\hat{T}_{12}$ through

$$\hat{T}_{12} = \lambda_s K_s \hat{\mu}(K_s^2; \lambda_s) = \lambda_s K_s (\lambda_s \hat{\beta}_1 - \hat{\beta}_{-1}), \tag{12}$$

where $\hat{\beta}_1$ and $\hat{\beta}_{-1}$ are functions of $\lambda_s$ and $K_s$ alone, and hence these are constants. We also recall that the recovery phenomenon is a decay process marked by a decreasing amount of shear $K(t)$ from an initially deformed state following a sudden reduction in the applied shearing force. In particular, if the process begins from the ultimate static state determined by eqn (12) and the load is reduced to zero, we expect that the shearing recovery $K(t)$ will decrease asymptotically from $K_s$ to zero. Since creep is an irreversible process in which energy is dissipated, this ideal recovery effect may never actually happen in a real material. Creep and recovery, however, usually are fairly slow motions and the material of differential type in eqn (1) behaves in static problems like an elastic material. Hence, when the load is reduced to zero, the material returns to its former state which is, in this example, its longitudinally prestretched state.

The governing equations for creep and recovery can be obtained from eqn (7). We find for creep

$$\eta \dot{K} = \hat{T}_{12} - \lambda_s K \hat{\mu}(K^2; \lambda_s), \tag{13}$$

and for recovery

$$\eta \dot{K} = -\lambda_s K \hat{\mu}(K^2; \lambda_s). \tag{14}$$

For a viscoelastic quadratic material, it may be shown from eqns (5), (6) and (8) that

$$\hat{\mu}(K^2; \lambda_s) = q_0 + q_1 K^2, \tag{15}$$

where $q_0$ and $q_1$ are material constants given by

$$q_0 = 2(\lambda_s C_1 + C_2) + (4\lambda_s C_3 + 2C_5)(\lambda_s^2 + 2\lambda_s^{-1} - 3) + (4C_4 + 2\lambda_s C_5)(\lambda_s^{-2} + 2\lambda_s - 3) \quad (16)$$

$$q_1 = (4\lambda_s C_3 + 2C_5)\lambda_s^2 + (4C_4 + 2\lambda_s C_5)\lambda_s. \quad (17)$$

In view of eqn (10), it follows from eqn (15) that for all $\lambda_s > 0$ and all $K \in (-\infty, \infty)$,

$$q_0 = \hat{\mu}(0; \lambda_s) > 0 \quad \text{and} \quad q_1 \geqslant 0. \quad (18)$$

Moreover, it is seen from eqn (15) that

$$\hat{\mu}(K^2; 1) = \mu_0 + 2\mu_1 K^2, \quad (19)$$

in which

$$\mu_0 = \lim_{\lambda_s \to 1} q_0 = 2(C_1 + C_2) > 0 \quad \text{and} \quad 2\mu_1 \equiv \lim_{\lambda_s \to 1} q_1 = 4(C_3 + C_4 + C_5) \geqslant 0. \quad (20)$$

3.1.1. *Creep shearing of a quadratic material of differential type.* Returning to the creep shearing relation (13) for a viscoelastic quadratic material of differential type in a simple shear superimposed on a static uniaxial stretch, we introduce eqns (12) and (15) to obtain the shear deflection rate, called the creep rate (or creep speed), namely,

$$\dot{K} = \frac{1}{t_r}[(K_s - K) + \beta(K_s^3 - K^3)] = \frac{K_s - K}{t_r}[1 + \beta(K_s^2 + K_s K + K^2)], \quad (21)$$

where

$$\beta \equiv \frac{q_1}{q_0}, \quad t_r \equiv \frac{\eta}{\lambda_s q_0}. \quad (22)$$

The material ratio $\beta$ is a measure of the nonlinearity of the material, and $t_r$ is recognized as the retardation time, a common physical parameter that measures the effect of the speed of the creep and recovery processes. It is seen from eqn (18) that $\beta \geqslant 0$ and $t_r \geqslant 0$, as may be expected. When $C_3$, $C_4$ and $C_5$ vanish, from eqn (17), we have $q_1 = 0$ and $\beta = 0$. In this case our viscohyperelastic quadratic material (4) reduces to the viscoelastic Mooney–Rivlin model. The strength of the material nonlinearity increases with $\beta$.

Integration of eqn (21) from the initial horizontal configuration of the shear mountings where $K(0) = 0$ yields the travel time of the load $M$ during the creep process. We thus find

$$\frac{t}{t_r} = \frac{1}{1 + 3\beta K_s^2}\left[\begin{array}{c} \ln\left(\dfrac{K_s\sqrt{\beta K^2 + \beta K_s K + 1 + \beta K_s^2}}{(K_s - K)\sqrt{1 + \beta K_s^2}}\right) \\ + \dfrac{3\beta K_s}{\sqrt{4\beta + 3\beta^2 K_s^2}} \tan^{-1}\left(\dfrac{K\sqrt{4\beta + 3\beta^2 K_s^2}}{\beta K_s K + 2 + 2\beta K_s^2}\right) \end{array}\right]. \quad (23)$$

The exact solution of eqn (21) for the viscoelastic Mooney–Rivlin model for which $\beta = 0$, is given by the classical rule

$$K = K_s\left[1 - \exp\left(-\frac{t}{t_r}\right)\right], \quad (24)$$

a result which also follows easily from eqn (23). This solution, as remarked by Beatty and Zhou (1991), is a universal solution which is independent of the material constants. Accordingly, any two viscoelastic Mooney–Rivlin materials having the same retardation time will exhibit the same creep response ratio $K/K_s$.

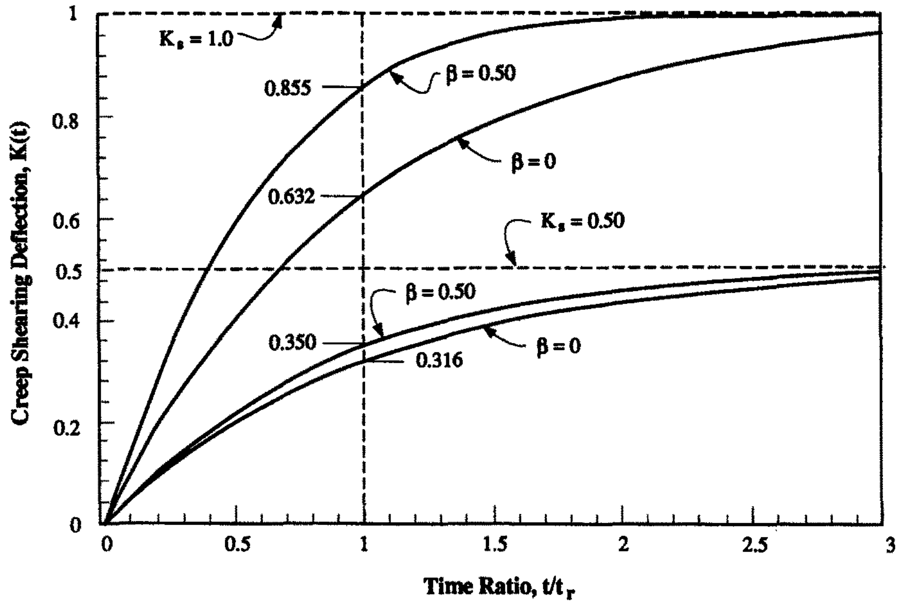



Fig. 2. Creep shearing response of a viscoelastic quadratic material of differential type in simple shear for selected values of the ultimate static shear deflection $K_s$ and the elastic material parameter $\beta$.

The creep shearing deflection $K(t)$, described by eqn (23), is illustrated in Fig. 2 for two finite, ultimate equilibrium shear deflections $K_s = 0.5$ and $K_s = 1$ of a viscoelastic quadratic material for which $\beta = 0.5$. The creep shearing deflection begins at $K = 0$ and approaches the ultimate static equilibrium position $K_s$ as $\dot{K} \to 0$ with $t \to \infty$, as anticipated. The corresponding solutions (24) for the Mooney–Rivlin model for which $\beta = 0$ are also shown in Fig. 2. It is seen that the creep rate (21), as shown by the increasing slope of the response curves in Fig. 2, increases with increasing values of $\beta$. This is also evident from the creep response ratio $K/K_s$ at $t = t_r$. This ratio for a few values of $\beta$ and $K_s$ is given in Table 1.

Thus, when $\beta = 0.5$ and $K_s = 0.5$, for example, 70.0% of the creep process has been completed by the retardation time $t = t_r$; and this grows to 85.5% when $K_s = 1$, as shown in Fig. 2. Equation (24) yields the universal creep response ratio of 63.2% for all Mooney–Rivlin materials in simple shear, as reported by Beatty and Zhou (1991).

The retardation time for the class of viscoelastic Mooney–Rivlin materials is derived by Beatty and Zhou (1991). The same result, however, may be obtained easily from eqn (22) for the class of viscoelastic quadratic materials in the special case when $C_3 = C_4 = C_5 = 0$. Thus, upon introducing $\alpha \equiv C_2/C_1$, we see from eqns (16) and $(20)_1$ that for the Mooney–Rivlin model

$$q_0 = \frac{\mu_0(\lambda_s + \alpha)}{1+\alpha}, \quad 2C_1 = \frac{\mu_0}{1+\alpha}, \quad 2C_2 = \frac{\alpha\mu_0}{1+\alpha}. \tag{25}$$

In consequence, from eqn $(22)_2$, the retardation time for a viscoelastic Mooney–Rivlin

Table 1. Creep response ratio $K/K_s$ at the retardation time, for selected values of $\beta$ and $K_s$

| | $K_s = 0.25$ | $K_s = 0.50$ | $K_s = 0.75$ | $K_s = 1$ |
|---|---|---|---|---|
| $\beta = 0$ | 63.2% | 63.2% | 63.2% | 63.2% |
| $\beta = 0.25$ | 64.1% | 66.7% | 70.8% | 76.0% |
| $\beta = 0.50$ | 65.0% | 70.0% | 77.4% | 85.5% |
| $\beta = 0.75$ | 65.8% | 73.1% | 82.9% | 91.9% |

material in a simple shear superimposed on an uniaxial stretch, as shown by Beatty and Zhou (1991), is given by

$$t_r = \frac{\eta(1+\alpha)}{\lambda_s \mu_0 (\lambda_s + \alpha)}. \tag{26}$$

Results for the neo-Hookean material are obtained for $\alpha = 0$.

3.1.2. *Shearing recovery for a quadratic material of differential type.* The shearing recovery process for a general isotropic, viscoelastic material of differential type whose current configuration is an ultimate static simple shear of amount $K_s$ superimposed on an uniaxial stretch is governed by eqn (14). We thus suppose that the shearing recovery begins from an initial static state $K_s$ when the applied external shearing load is suddenly removed. In particular, for a quadratic material of differential type, use of eqns (15) and (22) in eqn (14) yields the shearing recovery rate or speed

$$\dot{K} = -\frac{K}{t_r}(1+\beta K^2). \tag{27}$$

Thus, the recovery speed decreases from its initial value at $K(0) = K_s$ and ultimately vanishes when $K = 0$ as $t \to \infty$. The solution of eqn (27) is given by

$$K = \frac{K_s \exp(-t/t_r)}{\sqrt{1+\beta K_s^2(1-\exp(-2t/t_r))}}. \tag{28}$$

This solution completely describes the recovery process and also demonstrates the nonlinear effect of $\beta$. The shear recovery in eqn (28) is a decreasing function of $t/t_r$. Moreover, increasing the shear stiffness $\beta$ in eqn (28) decreases the recovery value of $K$ at a fixed value $t/t_r$ and, as seen in eqn (27), thus speeds the recovery process. The effect on the recovery speed due to $\beta$, however, is small compared with its effect on the creep speed. This is illustrated in Fig. 3 for $K_s = 1$ and 0.5, which may be compared with the corresponding cases for creep in Fig. 2. It is seen in Fig. 3 that the effect of $\beta$ on the recovery response diminishes as the amount of static shear $K_s$ decreases. Indeed, for small $K_s$ in eqn (28), we have $K = K_s \exp(-t/t_r)$, very nearly, and hence for small $K_s$, the material nonlinearity $\beta$ has essentially no effect on the shearing recovery. The recovery response ratio $(K_s - K)/K_s$ for $t = t_r$ is given in Table 2 for a few values of $\beta$ and $K_s$.

Thus, when $\beta = 0.5$ and $K_s = 0.5$, for example, 65.1% of the recovery process has been completed by the retardation time $t = t_r$. This value, however, does not differ significantly from the universal recovery response ratio of 63.2% for a Mooney–Rivlin material for which $\beta = 0$. Although neither the creep solution (23) nor the recovery solution (28) exhibit explicit dependence on the uniaxial initial stretch $\lambda_s$, notice that the static stretch affects the solution indirectly through eqn (22). Both $\beta$ and $t_r$ are functions of $\lambda_s$ through $q_0$ and $q_1$ given by eqns (16) and (17).

This concludes our study of the creep and recovery processes for a viscoelastic quadratic material of differential type (1). The same processes for a quadratic material of the rate type are explored next.

3.2. *Stress relaxation, creep and recovery of materials of rate type*

It is shown by Zhou (1991b) that the stress relaxation process for a general viscoelastic material of the rate type is characterized by the solution

$$\mathbf{T} = \hat{\mathbf{T}} + [\mathbf{T}_0 - \hat{\mathbf{T}}] \exp(-t/\xi) \tag{29}$$

for all Cauchy stress tensors $\mathbf{T}$. We recall that $\xi$ is the positive material time constant in eqn (2); otherwise, eqn (29) is independent of both the deformation parameters and the

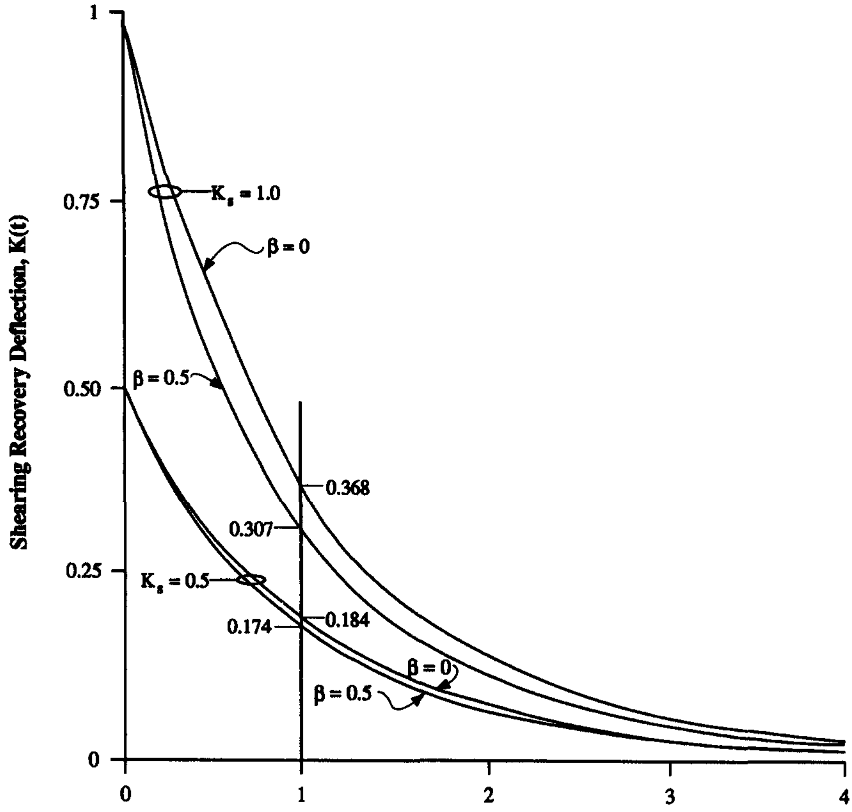



Fig. 3. Shearing recovery response of a viscoelastic quadratic material in simple shear for selected values of the initial static shear deflection $K_s$ and the elastic material parameter $\beta$.

elastic response functions, and in this sense may be regarded as universal [see e.g. Zhou (1991b)]. Thus, eqn (29) describes a universal stress relaxation process that begins from a certain initial stress $\mathbf{T}_0$ and relaxes exponentially to the ultimate equilibrium state on which the static stress $\hat{\mathbf{T}}$ is determined by the elastic part of eqn (2):

$$\hat{\mathbf{T}} = -\hat{p}\mathbf{1}+\hat{\beta}_1\hat{\mathbf{B}}+\hat{\beta}_{-1}\hat{\mathbf{B}}^{-1}, \tag{30}$$

where a circumflex denotes values at the ultimate equilibrium state. Implicit in the derivation of eqn (29) is the condition that the arbitrary pressure in eqn (2) is chosen so that $p(x,t) = \hat{p}(\hat{x})$ for all time in the stress relaxation process. For the system shown in Fig. 1, the only non-trivial, initial stress component is the uniaxial stress component

$$T^0_{22} = \left(\lambda_s - \frac{1}{\lambda_s^2}\right)\hat{\mu}_0(\lambda_s), \tag{31}$$

where $\hat{\mu}_0(\lambda_s) \equiv \hat{\mu}(0\,;\lambda_s)$ and we recall eqn (8).

Table 2. Recovery response ratio $(K_s - K)/K_s$ at the retardation time, for selected values of $\beta$ and $K_s$

| | $K_s = 0.25$ | $K_s = 0.50$ | $K_s = 0.75$ | $K_s = 1$ |
|---|---|---|---|---|
| $\beta = 0$ | 63.2% | 63.2% | 63.2% | 63.2% |
| $\beta = 0.25$ | 63.5% | 64.2% | 65.3% | 66.6% |
| $\beta = 0.50$ | 63.7% | 65.1% | 67.0% | 69.3% |
| $\beta = 0.75$ | 63.9% | 65.9% | 68.5% | 71.3% |

It is seen in eqn (29) that the material time constant $\xi$ is the retardation time:

$$t_r \equiv \xi, \tag{32}$$

which is the material time required for the stress to relax from its initial value $\mathbf{T}_0$ to 63.2% of its ultimate equilibrium value $\hat{\mathbf{T}}$. That is, by eqn (29), in evident physical component notation at $t = t_r$,

$$\frac{T_{ij}^0 - T_{ij}(t_r)}{T_{ij}^0 - \hat{T}_{ij}} = 1 - e^{-1} \approx 0.632. \tag{33}$$

Therefore, 63.2% of the total stress relaxation process is accomplished in time $t_r$. Clearly, if $\xi$ is small, for example, this occurs in a short time, hence the material time constant is a measure of the stress relaxation speed. Although it takes an infinitely long time to reach the ultimate equilibrium state, the great portion of the process is accomplished in the relatively short time (32). The universal constant (33) was noted by Zhou (1991b) and is characteristic of the stress relaxation process of all viscoelastic materials of rate type (2). Hence, it holds specifically for all quadratic materials of the rate type.

For the shear suspension system in Fig. 1, it is shown by Zhou (1991b) that the Cauchy normal and shear stress components in a simple shearing deformation of a viscoelastic material of the rate type (2) are related by

$$\xi \dot{T}_{11} + T_{11} = \beta_1 K^2 \lambda_s^2 \tag{34}$$

$$\xi \dot{T}_{12} + T_{12} = \lambda_s K \hat{\mu}(K^2; \lambda_s) + \dot{K}(\eta - \xi T_{11}) \tag{35}$$

$$\xi \dot{T}_{22} + T_{22} = (\lambda_s - \lambda_s^{-2}) \hat{\mu}(K^2; \lambda_s) + \beta_{-1} \lambda_s K^2 - 2\xi \dot{K} T_{12}. \tag{36}$$

The latter results require that the plane of shear shall be traction free so that $T_{33}(t) = T_{13}(t) = T_{23}(t) = 0$. These are a coupled system of ordinary differential equations for which we can offer no general solution. Nevertheless, it is useful to record formulae for specific processes to correct previous errors in Zhou (1991b).

3.2.1. *Creep shearing in a viscoelastic material of rate type.* The creep process is characterized by the growth of $K(t)$ under a constant applied shear stress $T_{12} = \hat{T}_{12}$, say. When the stress is produced by the effective weight of the load, $\hat{T}_{12}$ is provided by the equilibrium equation $2A\hat{T}_{12}/\lambda_s = Mg_0$. The remaining stress components, however, must vary with the amount of shear $K(t)$, and hence with time $t$. In any event, $T_{22}(t)$ is balanced by the symmetry of the supports of the spring-mass system shown in Fig. 1, and $T_{11}(t)$ must be known in order to determine the amount of shear $K(t)$ from eqn (35). If the load is released when $K = 0$, the shear will increase asymptotically to an ultimate equilibrium state defined by $\dot{K}(t) \to 0$ and $K(t) \to K_s$ as $t \to \infty$. Hence, from eqns (34), (35), (36) and upon recalling eqn (12), we find that the ultimate static stress components $\hat{T}_{11}$, $\hat{T}_{12}$ and $\hat{T}_{22}$ are related to the equilibrium shear deflection $K_s$ through

$$\begin{aligned} \hat{T}_{11} &= \hat{\beta}_1 K_s^2 \lambda_s^2 \\ \hat{T}_{12} &= \lambda_s K_s \hat{\mu}(K_s^2; \lambda_s) \\ \hat{T}_{22} &= (\lambda_s - \lambda_s^{-2}) \hat{\mu}(K_s^2; \lambda_s) + \hat{\beta}_{-1} K_s^2 \lambda_s, \end{aligned} \tag{37}$$

and from eqns (34), (35) and (36), we thus obtain the governing equations for the creep process in a viscoelastic material of the rate type:

$$\xi \dot{T}_{11} + T_{11} = \beta_1 K^2 \lambda_s^2$$

$$\xi \dot{T}_{22} + T_{22} = (\lambda_s - \lambda_s^{-2})\hat{\mu}(K^2 ; \lambda_s) + \beta_{-1}\lambda_s K^2 - 2\xi \dot{K}\hat{T}_{12} \tag{38}$$

$$\dot{K}(\eta - \xi T_{11}) = \hat{T}_{12}\lambda_s K \hat{\mu}(K^2 ; \lambda_s). \tag{39}$$

3.2.2. *Shearing recovery in a viscoelastic material of rate type.* We recall that the recovery phenomenon is a decay process marked by decreasing shear $K(t)$ from an initially deformed state following a sudden reduction in the applied shearing force. In particular, if the process begins from the static state determined by eqn (37), when the applied shearing load is reduced to zero so that $T_{12}(t) = 0$, the recovery shear $K(t)$ must decrease asymptotically from $K_s$ to zero. The governing equations for recovery are obtained from eqns (34), (35) and (36). We thus find for recovery in a simple shearing of a viscoelastic material of the rate type,

$$\xi \dot{T}_{11} + T_{11} = \beta_1 K^2 \lambda_s^2$$

$$\xi \dot{T}_{22} + T_{22} = (\lambda_s - \lambda_s^{-2})\hat{\mu}(K^2 ; \lambda_s) + \beta_{-1}\lambda_s K^2 \tag{40}$$

$$\dot{K}(\eta - \xi T_{11}) = -\lambda_s K \hat{\mu}(K^2 ; \lambda_s). \tag{41}$$

The recovery equation (41) in a simple shear of rate type material, contrary to the assertion by Zhou (1991b), is not the same as the corresponding relation (14) for a material of the differential type. Indeed, it is evident from the foregoing equations for the creep and recovery processes that a simple shearing of a material of rate type exhibits no useful simplifications for even the simplest neo-Hookean model for which $\beta_1 = \mu_0$, $\beta_{-1} = 0$. The unfortunate error in Zhou (1991b) derives from the improper assumption that $T_{11}$ may be constant. In consequence, the results for the simple shear deformation presented by Zhou (1991b) for materials of rate type are incorrect. The best we may expect is that the foregoing equations may be studied numerically, but we shall not pursue this here.

3.2.3. *Concluding remarks on stress relaxation in a viscoelastic material of rate type.* Returning briefly to the stress relaxation relation (33), we see that at the retardation instant the nonzero stress components in a simple shear superimposed on an uniaxial stretch are given by

$$T_{11}(t_r) = 0.632\hat{T}_{11}, \quad T_{12}(t_r) = 0.632\hat{T}_{12}, \quad T_{22}^0 - T_{22} = 0.632(T_{22}^0 - \hat{T}_{22}). \tag{42}$$

Here we recall eqn (31), the only nonzero initial stress component, and eqn (37). These are valid for all viscoelastic materials of the rate type.

This concludes our study of the stress relaxation, creep and recovery processes for a viscoelastic material of rate type in simple shear. In the next section we investigate the effect of viscous damping in the free oscillatory motion of a body supported by simple shear mountings shown in Fig. 1.

## 4. FINITE AMPLITUDE, FREE, DAMPED VIBRATIONS OF A SIMPLE SHEARING OSCILLATOR

Shear mountings of various designs are used in a variety of engineering applications, including vehicular suspension supports, machine and building foundation springs, and packaging supports. Of course, the physical nature of the vibrational motion of a load supported by shear mountings in any sort of application depends on their material characteristics. The mechanical behavior of a system with linear shear response, both with and without damping, certainly is well known. In addition, the problem of the free, undamped vibration of a body supported by simple shear springs characterized by a quadratic shear response function for a general compressible or incompressible, isotropic, hyperelastic material has been investigated by Beatty (1984). The more general undamped, finite amplitude, periodic motion of a rigid body supported by homogeneous, simple shear mountings of arbitrary design and material subsequently was addressed by Beatty (1988). In the latter case, a simple monotonicity condition on the shear response function sufficient for periodicity, and hence stability of the motion for arbitrary initial conditions, is provided. This general analysis for an arbitrary isotropic elastic material was applied by Beatty (1989) to study the stability of the motion of a body supported by a simple vehicular shear suspension system. Special cases were examined later by Beatty and Bhattacharyya (1989) and Bhattacharyya (1989) for the free and forced vibrational motion of a load supported by a quadratic material model. Viscoelastic effects, however, are not included in any of these studies. To examine the effect of damping in vibrational problems, Beatty and Zhou (1991) introduced the viscoelastic material of differential type (1); they applied this model to study the finite amplitude vibrations of a body supported by viscoelastic Mooney–Rivlin shear springs. The exact solution is given in terms of exponential and sinusoidal functions. In this section, the problem described earlier in Fig. 1 is studied for an incompressible, viscoelastic quadratic material of differential type. It will be shown that the motion of the body is governed by a damped Duffing equation. The averaging method is used to obtain an approximate solution for the free, damped oscillatory motion of the load.

To begin, we recall the engineering stress tensor $\mathbf{S} = \mathbf{T}\mathbf{F}^{-T}$ for an incompressible material [see e.g. Beatty (1987)]; then the engineering shear stress $S_{12} = T_{12}/\lambda_s$. As usual, we shall ignore the inertia of the shear springs. We shall also neglect bending of the mounts and thus suppose, of course, that appropriate surface tractions are applied to the shear mountings to effect their ideal simple shearing. Then, with Fig. 1 in mind, the acceleration of the center of mass of the load is $\ddot{x} = L\lambda_s\ddot{K}$, in which $K(t) = \tan\gamma(t)$ as noted earlier. Hence the equation of motion of the load $M$ is

$$ML\lambda_s\ddot{K} = Mg_0 - 2AS_{12}. \tag{43}$$

With the aid of eqns (7), (15) and the aforementioned relation for the shear stress component, the equation of motion of the load for the viscoelastic spring-mass system of differential type becomes

$$\ddot{K} + 2\nu\dot{K} + \omega^2(K + \beta K^3) = p_0^2, \tag{44}$$

in which

$$\omega^2 \equiv \frac{2Aq_0}{ML\lambda_s}, \quad 2\nu \equiv \frac{2A\eta}{ML\lambda_s^2} = \omega^2 t_r, \quad p_0^2 \equiv \frac{g_0}{L\lambda_s} = \frac{g\sin\theta}{L\lambda_s}, \tag{45}$$

and $\beta$ and $t_r$ are defined in eqn (22). It is seen that when the quadratic material parameter $\beta \to 0$, eqn (44) reduces to eqn (40) in Beatty and Zhou (1991) for a viscoelastic Mooney–Rivlin oscillator for which the exact solution of the damped oscillation is well known. Otherwise, we recognize eqn (44) as a damped Duffing equation whose closed form solution is unknown. Therefore, we shall seek an approximate solution.

We first introduce the amount of shear relative to the equilibrium state, namely,

$$\kappa \equiv K - K_s, \tag{46}$$

in which the static shear deflection $K_s$, from eqn (44), is given by

$$K_s + \beta K_s^3 = \frac{p_0^2}{\omega^2} = \frac{Mg\sin\theta}{2Aq_0}. \tag{47}$$

Thus, with the aid of eqns (46) and (47), the equation of motion (44) may be written in the form

$$\ddot{\kappa}+2\nu\dot{\kappa}+\omega^2[\kappa+\beta(3K_s^2\kappa+3K_s\kappa^2+\kappa^3)] = 0. \tag{48}$$

This equation describes the finite amplitude, free vibrations of the load about the equilibrium state of a viscoelastic quadratic oscillator of differential type. Its exact solution is unknown. We are thus led to explore an approximate analysis, commonly known as the averaging method, to study the nature of the solution of eqn (48).

Following Hagedorn (1978) and originally Kryloff and Bogoliuboff (1947), we first rewrite eqn (48) in the form

$$\ddot{\kappa}+\omega^2\kappa = s(\kappa,\dot{\kappa}), \tag{49}$$

with

$$s(\kappa,\dot{\kappa}) \equiv -2\nu\dot{\kappa}-\omega^2\beta(3K_s^2\kappa+3K_s\kappa^2+\kappa^3). \tag{50}$$

The solution of eqn (49) for the case $s(\kappa,\dot{\kappa}) = 0$ is given by

$$\kappa = A_0 \sin(\omega t+\psi_0) \quad \text{hence} \quad \dot{\kappa} = \omega A_0 \cos(\omega t+\psi_0), \tag{51}$$

in which both the amplitude $A_0$ and the initial phase $\psi_0$ are constants determined by the initial data. We now consider the nonlinear equation (49) in which $s(\kappa,\dot{\kappa}) \neq 0$. In accordance with the averaging method [see e.g. Hagedorn (1978); Kryloff and Bogoliuboff (1947)], $A_0$ and $\psi_0$ in eqn (51) are replaced by functions $A(t)$ and $\psi(t)$, respectively. We then consider a solution of the form similar to eqn (51) so that

$$\kappa(t) = A(t)\sin[\omega t+\psi(t)] \tag{52}$$

and

$$\dot{\kappa}(t) = \omega A(t)\cos[\omega t+\psi(t)]. \tag{53}$$

Hence, differentiation of eqn (52) shows that eqn (53) holds if and only if

$$\dot{A}\sin\psi_1+\dot{\psi}A\cos\psi_1 = 0, \tag{54}$$

in which

$$\psi_1(t) \equiv \omega t+\psi(t). \tag{55}$$

Use of eqns (52), (53) and (55) in eqn (49) yields the additional relation

$$\dot{A}\cos\psi_1-\dot{\psi}A\sin\psi_1 = \frac{1}{\omega}s(A\sin\psi_1,\omega A\cos\psi_1). \tag{56}$$

Thus, eqns (54) and (56) deliver the following equations for $\dot{A}$ and $\dot{\psi}$:

$$\dot{A} = \frac{1}{\omega}s(A\sin\psi_1,\omega A\cos\psi_1)\cos\psi_1 \tag{57}$$

$$\dot{\psi} = -\frac{1}{\omega A}s(A\sin\psi_1,\omega A\cos\psi_1)\sin\psi_1. \tag{58}$$

We now assume that the function $s(\kappa,\dot{\kappa})$ is sufficiently small that both the amplitude and phase change only slowly. We can then simplify the differential equations (57) and (58) by replacing the right-hand side of these equations by their temporal mean values identified in the variable $\psi_1$ over the interval $[0, 2\pi]$, that is, in the period $2\pi/\omega$. In forming this mean value, $A$ and $\psi$ are held constant in the right-hand sides of eqns (57) and (58). Hence, we have

$$\dot{A} = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{\omega}s(A\sin\psi_1, \omega A\cos\psi_1)\cos\psi_1\right]\mathrm{d}\psi_1 \tag{59}$$

$$\dot{\psi} = -\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{\omega A}s(A\sin\psi_1, \omega A\cos\psi_1)\sin\psi_1\right]\mathrm{d}\psi_1. \tag{60}$$

The result is an autonomous system of equations that yields a first approximation to the time dependence of the amplitude and phase. Substitution of eqn (50) into eqns (59) and (60) and integration of the results delivers the following equations for $A$ and $\psi$:

$$\dot{A} = -\nu A \tag{61}$$

$$\dot{\psi} = \frac{3\omega\beta}{8}(4K_s^2 + A^2). \tag{62}$$

Finally, integration of these equations gives

$$A = A_0\,\mathrm{e}^{-\nu t} \tag{63}$$

$$\psi = \psi_0 + \frac{3\omega\beta}{8}\left(4K_s^2 t - \frac{A_0^2}{2\nu}\mathrm{e}^{-2\nu t}\right). \tag{64}$$

Hence, by eqns (46), (52), (63) and (64), the approximate solution to eqn (49), and hence to eqn (44), for finite amplitude, free, damped shearing oscillations of the load is given by

$$K = K_s + A_0\,\mathrm{e}^{-\nu t}\sin\left[\omega t(1 + \tfrac{3}{2}\beta K_s^2) - \frac{3\omega\beta A_0^2}{16\nu}\mathrm{e}^{-2\nu t} + \psi_0\right]. \tag{65}$$

Thus, with $K(0) = K_s + A_0$, we have

$$\psi_0 = \frac{\pi}{2} + \frac{3\omega\beta A_0^2}{16\nu},$$

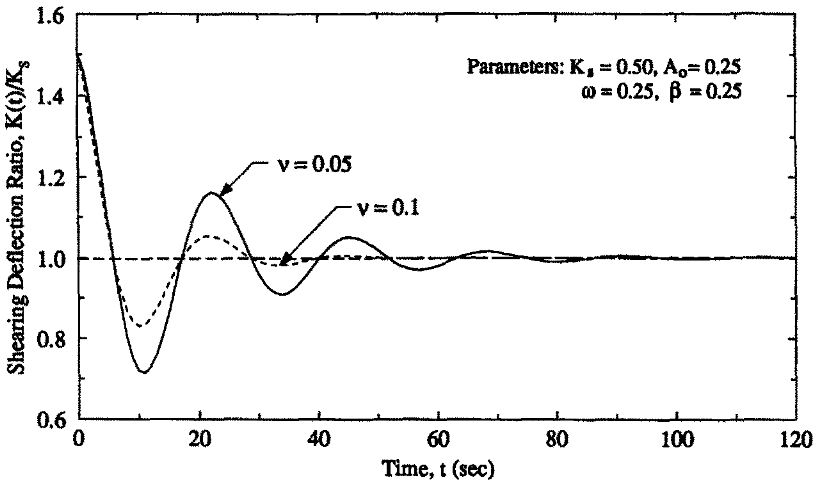
and hence

$$K = K_s + A_0\,\mathrm{e}^{-\nu t}\cos\left[\omega t(1 + \tfrac{3}{2}\beta K_s^2) + \frac{3\omega\beta A_0^2}{16\nu}(1 - \mathrm{e}^{-2\nu t})\right]. \tag{66}$$

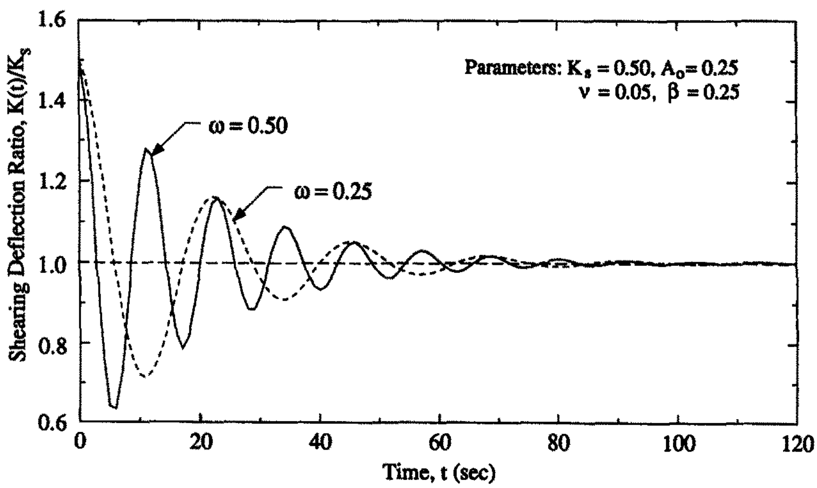
The solution (66), also illustrated in Fig. 4 for selected values of the several parameters, is a damped oscillatory motion whose amplitude $A_0$ decays exponentially. It is seen in eqn (66) that the frequency response depends on the amplitude, a characteristic typical of nonlinear motion. The effect of variation in the damping coefficient $\nu$ is shown in Fig. 4(a) and the effect of variation of the natural circular frequency $\omega$ is shown in Fig. 4(b). For a horizontal motion, the equilibrium equation (47) requires $K_s = 0$, and hence eqn (66) simplifies to

$$K = A_0\,\mathrm{e}^{-\nu t}\cos\left[\omega t + \frac{3\omega\beta A_0^2}{16\nu}(1 - \mathrm{e}^{-2\nu t})\right]. \tag{67}$$

When $\beta \to 0$, eqns (66) and (67) reduce to the classical solutions obtained previously by Beatty and Zhou (1991) for the Mooney–Rivlin oscillator.




(a)




(b)

Fig. 4. Shearing deflection ratio as a function of time, for (a) two values of the damping coefficient and fixed parameter values and (b) two values of the circular frequency and assigned parameter values.

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